数理解析研究所講究録
第 1948 巻 2015 年 124-136

ASYMPTOTIC ANALYSIS OF OSCILLATORY INTEGRALS WITH SMOOTH WEIGHTS.

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ABSTRACT. We announce some results obtained in [16], which is a joint work with Kamimoto, on the asymptotic behavior of oscillatory integrals with smooth weights. Our results show that the optimal rates of decay for weighted oscillatory integrals, whose phases and weights are contained in a certain class of smooth functions including the real analytic class, can be expressed by the Newton distance and multiplicity defined in terms of geometrical relationship of the Newton polyhedra of the phase and the weight.

1. INTRODUCTION

We consider the asymptotic behavior of oscillatory integrals of the weighted form

$$I(t;\varphi)=\int_{\mathbb{R}^n}e^{itf(x)}g(x)\varphi(x)dx \tag{1.1}$$

for large values of the real parameter t, where

- f is a real-valued smooth (C^∞) function defined on an open neighborhood U of the origin in $\mathbb{R}^n$, which is called the *phase*;
- g is a real-valued smooth function defined on U, which is called the *weight*;
- φ is a real-valued smooth function defined on $\mathbb{R}^n$ and the support of φ is contained in U. $g\varphi$ is called the *amplitude*.

The investigations of the behavior of $I(t;\varphi)$ as $t\to+\infty$ are very important subjects occurring in harmonic analysis, partial differential equations, probability theory, number theory, etc. We refer to [22] as a great exposition of such issues. There is no harm in assuming that $f(0)=0$ since one can always factor out $e^{itf(0)}$; If f has no critical point on the support of φ, then $I(t;\varphi)$ decays faster than t^{-N} for any positive integer N. Hence, in this article, we always assume that

$$f(0)=0 \quad \text{and} \quad \nabla f(0)=0.$$

When f has a nondegenerate critical point at the origin, then the asymptotic expansions of $I(t;\varphi)$ are precisely computed by using the Morse lemma and Fresnel integrals. (See Section 2.3, Chapter VIII in [22].) We are particularly interested in the degenerate phase case.

In the real analytic phase case, the following is shown (see [13],[17]) by using a famous Hironaka's resolution of singularities [8]: If f is real analytic and the support of φ is contained in a sufficiently small open neighborhood of the origin, then the integral $I(t;\varphi)$ has an asymptotic expansion of the form

$$I(t;\varphi) \sim \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha k}(\varphi) t^{\alpha} (\log t)^{k-1} \quad \text{as } t \to +\infty, \tag{1.2}$$

where α runs through a finite number of arithmetic progressions, not depending on the amplitude, which consist of negative rational numbers. In special cases of the smooth phase, $I(t;\varphi)$ also admits an asymptotic expansion of the same form as in (1.2) (see [21], [15] and Remark 3.2 in this article). In order to see the decay property of $I(t;\varphi)$, we are interested in the leading term of (1.2) and define the following index.

Definition 1.1. Let f, g be smooth functions, for which the oscillatory integral (1.1) admits the asymptotic expansion of the form (1.2). The set $S(f,g)$ consists of pairs (α, k) such that for each neighborhood of the origin in $\mathbb{R}^n$, there exists a smooth function φ with support contained in this neighborhood for which $C_{\alpha k}(\varphi) \neq 0$ in (1.2). The maximum element of the set $S(f,g)$, under the lexicographic ordering, is denoted by $(\beta(f,g), \eta(f,g))$, i.e., $\beta(f,g)$ is the maximum of values α for which we can find k so that (α, k) belongs to $S(f,g)$; $\eta(f,g)$ is the maximum of integers k satisfying that $(\beta(f,g), k)$ belongs to $S(f,g)$. We call $\beta(f,g)$ the *oscillation index* of (f,g) and $\eta(f,g)$ the *multiplicity* of its index.

Roughly speaking, the leading asymptotic behavior of $I(t;\varphi)$ is represented by using $\beta(f,g)$ and $\eta(f,g)$ as follows: There exists some smooth function φ defined on U such that

$$I(t;\varphi) \sim C(\varphi) t^{\beta(f,g)} (\log t)^{\eta(f,g)-1},$$

where $C(\varphi) \neq 0$. In the unweighted case, i.e., $g \equiv 1$, the multiplicity $\eta(f,1)$ is one less than the corresponding multiplicity in [1], p. 183.

The purpose of this article is to determine or precisely estimate the oscillation index and its multiplicity by means of appropriate information of the phase and the weight. In the unweighted case, many strong results have been obtained. In a seminal work of Varchenko [23] (see also [1]), the oscillation index and its multiplicity are investigated in detail in the case when the phase is real analytic and satisfies a certain nondegeneracy condition. In particular, they are determined or estimated by the geometrical data of the Newton polyhedron of the phase. (See Theorem 3.1 below.) In his analysis, some concrete resolution of singularities constructed from the theory of toric varieties based on the geometry of the Newton polyhedron of the phase plays an important role. (Recently, it is shown in [15] that the above result of Varchenko can be generalized to the case when the phase belongs to a wider class of smooth functions, denoted by $\hat{\mathcal{E}}(U)$, including the real analytic class. See Remark 3.2 below). On the other hand, another approach, which is inspired by the work of Phong and Stein on oscillatory integral operators in the seminal paper [19], has been developed and succeeds to give many strong results ([4],[5],[6],[9],[10],[3], etc.). In particular, the two-dimensional case has been deeply understood. In these papers, the importance of resolution of singularities constructed from the Newton polyhedron is strongly recognized.

Until now, there are not so many studies about the weighted case, but some precise results have been obtained in [24],[1],[20],[2],[18]. In these studies, the Newton polyhedra of both the phase and the weight play important roles. Particularly, in [24],[1],[2],[18], it was made an attempt to generalize the results of Varchenko in [23] as directly as possible in the weighted case under the nondegeneracy condition on the phase. Vassiliev [24] considers the case when the weight is a monomial. In [1], there are assertions related to oscillatory integrals with general smooth weights. Unfortunately, they does not hold and more additional assumptions are necessary to obtain corresponding assertions. Okada and Takeuchi [18] consider the case when the phase is *convenient*, i.e., the Newton polyhedron of the phase intersects all the coordinate axes. In [2], we generalize and improve the results of Varchenko, and particularly give several sufficient conditions to determine or precisely estimate the oscillation index and its multiplicity, which also include the results in [24],[18]. Pramanik and Yang [20] consider the two-dimensional case with the weight of the form $g(x) = |h(x)|^{\epsilon}$, where h is real analytic and ϵ is positive. (This g may not be smooth.) Their approach is based on not only the method of Varchenko but also the above-mentioned work of Phong and Stein [19]. As a result, they succeed to remove the nondegeneracy hypothesis on the phase. Recently, Greenblatt [7] also considers the asymptotic behavior of oscillatory integrals with nonsmooth weights.

Our new results are generalizations and improvements of the previous studies in [2], which generalizes the above-mentioned results of Varchenko [23] to the weighted case. As mentioned above, the importance of resolution of singularities has been strongly recognized in earlier successive investigations of the behavior of oscillatory integrals. Let us review our analysis from this point of view. The resolution in the work of Varchenko [23] is based on the theory of toric varieties. His method gives quantitative resolution by means of the geometry of the Newton polyhedron of the phase. In [15], we directly generalize this resolution to the class $\hat{\mathcal{E}}(U)$ of smooth functions. Furthermore, in order to consider the weighted case, some kind of *simultaneous* resolution of singularities with respect to two functions, i.e., the phase and the weight, must be constructed. From the viewpoint of the theory of toric varieties, simultaneous resolution of singularities reflects finer simplicial subdivision of a fan constructed from the Newton polyhedra of the above two functions. Therefore, it is essentially important to investigate accurate relationship between cones of this subdivided fan and faces of the Newton polyhedra of the two functions. This situation has been investigated in [2], but deeper understanding this relationship gives stronger results about the behavior of oscillatory integrals. In particular, we succeed to give explicit formulae of the coefficient of the leading term of the asymptotic expansion under some appropriate conditions, which reveals that the behavior of oscillatory integrals is decided by some important faces, which are called *principal faces* (see Definition 2.5 below), of the Newton polyhedra of the phase and the weight.

It is known (see, for instance, [12],[1]) that the asymptotic analysis of oscillatory integral (1.1) can be reduced to an investigation of the poles of the (weighted) *local zeta function*

$$Z(s;\varphi) = \int_{\mathbb{R}^n} |f(x)|^s g(x)\varphi(x)dx,$$

where f, g, φ are the same as in (1.1). The substantial analysis in our argument is to investigate properties of poles of the local zeta function $Z(s;\varphi)$ by means of the Newton polyhedra of the functions f and g.

Notation and symbols.

- We denote by $\mathbb{Z}_+, \mathbb{R}_+$ the subsets consisting of all nonnegative numbers in $\mathbb{Z}, \mathbb{R}$, respectively.
- We use the multi-index as follows. For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, define

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n,$$
$$x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \quad \partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$
$$\alpha! = \alpha_1!\cdots\alpha_n!, \quad 0! = 1.$$

- For $A, B \subset \mathbb{R}^n$ and $c \in \mathbb{R}$, we set

$$A + B = \{a + b \in \mathbb{R}^n : a \in A \text{ and } b \in B\}, \quad c\cdot A = \{ca \in \mathbb{R}^n : a \in A\}.$$

- We express by $\mathbf{1}$ the vector $(1, \dots, 1)$ or the set $\{(1, \dots, 1)\}$.

2. Preliminaries

2.1. **Polyhedra.** Let us explain fundamental notions in the theory of convex polyhedra, which are necessary for our investigation. Refer to [25] for general theory of convex polyhedra.

For $(a, l) \in \mathbb{R}^n \times \mathbb{R}$, let $H(a, l)$ and $H^+(a, l)$ be a hyperplane and a closed halfspace in $\mathbb{R}^n$ defined by

$$H(a, l) := \{x \in \mathbb{R}^n : \langle a, x\rangle = l\},$$
$$H^+(a, l) := \{x \in \mathbb{R}^n : \langle a, x\rangle \geq l\},$$

respectively. A (*convex rational*) *polyhedron* is an intersection of closed halfspaces: a set $P \subset \mathbb{R}^n$ presented in the form $P = \bigcap_{j=1}^N H^+(a^j, l_j)$ for some $a^1, \dots, a^N \in \mathbb{Z}^n$ and $l_1, \dots, l_N \in \mathbb{Z}$.

Let P be a polyhedron in $\mathbb{R}^n$. A pair $(a, l) \in \mathbb{Z}^n \times \mathbb{Z}$ is said to be *valid* for P if P is contained in $H^+(a, l)$. A *face* of P is any set of the form $F = P \cap H(a, l)$, where (a, l) is valid for P. Since $(0, 0)$ is always valid, we consider P itself as a trivial face of P; the other faces are called *proper faces*. Conversely, it is easy to see that any face is a polyhedron. Considering the valid pair $(0, -1)$, we see that the empty set is always a face of P. Indeed, $H^+(0, -1) = \mathbb{R}^n$, but $H(0, -1) = \emptyset$. We write

(2.1) $$\mathcal{F}[P] = \text{the set of all nonempty faces of } P.$$

The *dimension* of a face F is the dimension of its affine hull (i.e., the intersection of all affine flats that contain F), which is denoted by $\dim(F)$. The faces of dimensions $0, 1$ and $\dim(P) - 1$ are called *vertices*, *edges* and *facets*, respectively. The *boundary* of a polyhedron P, denoted by ∂P, is the union of all proper faces of P. For a face F, ∂F is similarly defined.

Every polyhedron treated in this article satisfies a condition in the following lemma.

Lemma 2.1. *Let $P \subset \mathbb{R}_+^n$ be a polyhedron. Then the following conditions are equivalent.*

(i) $P + \mathbb{R}_+^n \subset P$;

(ii) *There exists a finite set of pairs* $\{(a^j, l_j)\}_{j=1}^N \subset \mathbb{Z}_+^n \times \mathbb{Z}_+$ *such that* $P = \bigcap_{j=1}^N H^+(a^j, l_j)$.

2.2. **Newton polyhedra.** Let f be a smooth function defined on a neighborhood of the origin in $\mathbb{R}^n$, which has the Taylor series at the origin:

$$f(x) \sim \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha \qquad \text{with } c_\alpha = \frac{\partial^\alpha f(0)}{\alpha!}. \tag{2.2}$$

Definition 2.2. The *Newton polyhedron* $\Gamma_+(f)$ of f is defined to be the convex hull of the set $\bigcup\{\alpha + \mathbb{R}_+^n : c_\alpha \neq 0\}$.

It is known that the Newton polyhedron is a polyhedron (see [25]). The following classes of smooth functions often appear in this article.

- f is said to be *flat* if $\Gamma_+(f) = \emptyset$ (i.e., all derivatives of f vanish at the origin).
- f is said to be *convenient* if the Newton polyhedron $\Gamma_+(f)$ intersects all the coordinate axes.

2.3. **Newton distance and multiplicity.** Let f, g be nonflat smooth functions defined on a neighborhood of the origin in $\mathbb{R}^n$. We define the Newton distance and the Newton multiplicity with respect to the pair (f, g). At the same time, consider important faces of $\Gamma_+(f)$ and $\Gamma_+(g)$, which will initially affect the asymptotic behavior of oscillatory integrals. Hereafter, we assume that $f(0) = 0$.

Definition 2.3. The *Newton distance* of the pair (f, g) is defined by

$$d(f, g) := \max\{d > 0 : \partial\Gamma_+(f) \cap d \cdot (\Gamma_+(g) + \mathbf{1}) \neq \emptyset\}. \tag{2.3}$$

This distance will be crucial to determine or estimate the oscillation index. In [1], p.254, the number $d(f, g)$ is called the *coefficient of inscription* of $\Gamma_+(g)$ in $\Gamma_+(f)$. (In [1], this number is defined by $\min\{d > 0 : d \cdot \Gamma_+(g) \subset \Gamma_+(f)\}$, which must be corrected as in (2.3).)

We define the map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Phi(\beta) := d(f, g)(\beta + \mathbf{1}).$$

The image of $\Gamma_+(g)$ by the map Φ comes in contact with the boundary of $\Gamma_+(f)$. We denote by $\Gamma_0(f)$ this contacting set on $\partial\Gamma_+(f)$ and by $\Gamma_0(g)$ the image of $\Gamma_0(f)$ by the inverse map of Φ, i.e.,

$$\Gamma_0(f) := \partial\Gamma_+(f) \cap \Phi(\Gamma_+(g)) \; (= \partial\Gamma_+(f) \cap d(f,g) \cdot (\Gamma_+(g) + \mathbf{1}));$$
$$\Gamma_0(g) := \Phi^{-1}(\Gamma_0(f)) \; \left(= \left(\frac{1}{d(f,g)} \cdot \partial\Gamma_+(f) - \mathbf{1} \right) \cap \Gamma_+(g) \right).$$

Note that $\Gamma_0(g)$ is a certain union of faces of $\Gamma_+(g)$.

Let us define the Newton multiplicity and important faces of $\Gamma_+(f)$ and $\Gamma_+(g)$, which will play important roles in the investigation of multiplicity of the oscillation index. We defince the map

$$\tau_f : \partial\Gamma_+(f) \to \mathcal{F}[\Gamma_+(f)]$$

as follows (see the definition (2.1) of $\mathcal{F}[\cdot]$): For $\alpha \in \partial\Gamma_+(f)$, let $\tau_f(\alpha)$ be the smallest face of $\Gamma_+(f)$ containing α. In other words, $\tau_f(\alpha)$ is the face whose relative interior contains the point $\alpha \in \partial\Gamma_+(f)$. Define

$$\mathcal{F}_0[\Gamma_+(f)] := \{\tau_f(\alpha) : \alpha \in \Gamma_0(f)\} (\subset \mathcal{F}[\Gamma_+(f)]).$$

Definition 2.4. The *Newton multiplicity* of the pair (f,g) is defined by

$$m(f,g) := \max\{n - \dim(\tau) : \tau \in \mathcal{F}_0[\Gamma_+(f)]\}.$$

Definition 2.5. Define

$$\mathcal{F}_*[\Gamma_+(f)] := \{\tau \in \mathcal{F}_0[\Gamma_+(f)] : n - \dim(\tau) = m(f,g)\}.$$

The elements of the above set are called the *principal faces of* $\Gamma_+(f)$. Define

$$\mathcal{F}_*[\Gamma_+(g)] := \{\Phi^{-1}(\tau) \cap \Gamma_+(g) : \tau \in \mathcal{F}_*[\Gamma_+(f)]\}.$$

It is easy to see that every element of the above set is a face of $\Gamma_+(g)$, which is called a *principal face of* $\Gamma_+(g)$. The map $\Psi : \mathcal{F}_*[\Gamma_+(f)] \to \mathcal{F}_*[\Gamma_+(g)]$ is defined as $\Psi(\tau) := \Phi^{-1}(\tau) \cap \Gamma_+(g)$. It is easy to see that this map is bijective. We say that $\tau \in \mathcal{F}_*[\Gamma_+(f)]$ (resp. $\gamma \in \mathcal{F}_*[\Gamma_+(g)]$) *is associated to* $\gamma \in \mathcal{F}_*[\Gamma_+(g)]$ (resp. $\tau \in \mathcal{F}_*[\Gamma_+(f)]$), if $\gamma = \Psi(\tau)$.

Remark 2.6. In [2], the union of the faces belonging to $\mathcal{F}_*[\Gamma_+(g)]$ was called the *essential set* on $\Gamma_0(g)$. It is shown in [2] that every two faces belonging to $\mathcal{F}_*[\Gamma_+(g)]$ are disjoint.

Remark 2.7. Let us consider the case $g(0) \neq 0$. Then $\Gamma_+(g) = \mathbb{R}^n_+$. In this case, since $d(f,g)$ and $m(f,g)$ are independent of g, we simply denote them by $d(f)$ and $m(f)$, respectively. It is easy to see the following:

- $d(f,g) \leq d(f)$ for general g;
- The Newton distance $d(f)$ is determined by the point $q_* = (d(f), \ldots, d(f))$, which is the intersection of the line $\alpha_1 = \cdots = \alpha_n$ with $\partial\Gamma_+(f)$;
- The principal face τ_* of $\Gamma_+(f)$ is the smallest face of $\Gamma_+(f)$ containing the point q_*;
- $m(f) = n - \dim(\tau_*)$.

2.4. **The γ-part.** Let f be a smooth function defined on a neighborhood V of the origin whose Taylor series at the origin is as in (2.2), $P \subset \mathbb{R}_+^n$ a nonempty polyhedron in $\mathbb{R}_+^n$ containing $\Gamma_+(f)$ and γ a face of P. Note that P satisfies the condition: $P + \mathbb{R}_+^n \subset P$ (see Lemma 2.1).

Definition 2.8. We say that f *admits the γ-part* on an open neighborhood $U \subset V$ of the origin if for any x in U the limit:

$$\lim_{t \to 0} \frac{f(t^{a_1}x_1, \ldots, t^{a_n}x_n)}{t^l} \tag{2.4}$$

exists for *all* valid pairs $(a, l) = ((a_1, \ldots, a_n), l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ defining γ. When f admits the γ-part, it is known in [15], Proposition 5.2 (iii), that the above limits take the same value for any (a, l), which is denoted by $f_\gamma(x)$. We consider f_γ as a function on U, which is called the γ-part of f on U.

Remark 2.9. We summarize important properties of the γ-part. See [15] for the details.

(i) The γ-part f_γ is a smooth function defined on U.

(ii) If f admits the γ-part f_γ on U, then f_γ has the quasihomogeneous property:

$$f_\gamma(t^{a_1}x_1, \ldots, t^{a_n}x_n) = t^l f_\gamma(x) \quad \text{for} \quad 0 < t < 1 \text{ and } x \in U,$$

where $(a, l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ is a valid pair defining γ.

(iii) For a compact face γ of $\Gamma_+(f)$, f always admits the γ-part near the origin and $f_\gamma(x)$ equals the polynomial $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$, which is the same as the well-known γ-part of f in [23],[1]. Note that γ is a compact face if and only if every valid pair $(a, l) = (a_1, \ldots, a_n)$ defining γ satisfies $a_j > 0$ for any j.

(iv) Let f be a smooth function and γ a noncompact face of $\Gamma_+(f)$. Then, f does not admit the γ-part in general. If f admits the γ-part, then the Taylor series of $f_\gamma(x)$ at the origin is $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$, where the Taylor series of f is as in (2.2).

(v) Let f be a smooth function and γ a face defined by the intersection of $\Gamma_+(f)$ and some coordinate hyperplane. Altough γ is a noncompact face if $\gamma \neq \emptyset$, f always admits the γ-part. Indeed, for every valid pair (a, l) defining γ, we have $l = 0$, which implies the existence of the limit (2.4).

(vi) If f is real analytic and γ is a face of $\Gamma_+(f)$, then f admits the γ-part. Moreover, $f_\gamma(x)$ is real analytic and is equal to a convergent power series $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$ on some neighborhood of the origin.

2.5. **The classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$.** Let P be a polyhedron (possibly an empty set) in $\mathbb{R}^n$ satisfying $P + \mathbb{R}_+^n \subset P$ when $P \neq \emptyset$. Let U be an open neighborhood of the origin.

Definition 2.10. Denote by $\mathcal{E}[P](U)$ the set of smooth functions on U whose Newton polyhedra are contained in P. Moreover, when $P \neq \emptyset$, we denote by $\hat{\mathcal{E}}[P](U)$ the set of the elements f in $\mathcal{E}[P](U)$ such that f admits the γ-part on some neighborhood of the origin for any face γ of P. When $P = \emptyset$, $\hat{\mathcal{E}}[P](U)$ is defined to be the set $\{0\}$, i.e., the set consisting of only the function identically equaling zero on U.

We summarize properties of the classes $\mathcal{E}[P](U)$ and $\hat{\mathcal{E}}[P](U)$, which can be directly seen from their definitions:

(i) $\hat{\mathcal{E}}[\mathbb{R}_+^n](U) = \mathcal{E}[\mathbb{R}_+^n](U) = C^\infty(U)$;

(ii) If P_1, $P_2 \subset \mathbb{R}_+^n$ are polyhedra with $P_1 \subset P_2$, then $\mathcal{E}[P_1](U) \subset \mathcal{E}[P_2](U)$ and $\hat{\mathcal{E}}[P_1](U) \subset \hat{\mathcal{E}}[P_2](U)$;

(iii) $(C^\omega(U) \cap \mathcal{E}[P](U)) \subsetneq \hat{\mathcal{E}}[P](U) \subsetneq \mathcal{E}[P](U)$;

(iv) $\mathcal{E}[P](U)$ and $\hat{\mathcal{E}}[P](U)$ are $C^\infty(U)$-modules and ideals of $C^\infty(U)$.

Definition 2.11. $\hat{\mathcal{E}}(U) := \{f \in C^\infty(U) : f \in \hat{\mathcal{E}}[\Gamma_+(f)](U)\}$.

It is easy to see the following properties of the class $\hat{\mathcal{E}}(U)$.

(i) $C^\omega(U) \subsetneq \hat{\mathcal{E}}(U) \subsetneq C^\infty(U)$;

(ii) When f is flat but $f \not\equiv 0$, f does not belong to $\hat{\mathcal{E}}(U)$.

The class $\hat{\mathcal{E}}(U)$ contains many kinds of smooth functions.

- $\hat{\mathcal{E}}(U)$ contains the function identically equaling zero on U.
- Every real analytic function defined on U belongs to $\hat{\mathcal{E}}(U)$. (From (vi) in Remark 2.9.)
- If $f \in C^\infty(U)$ is convenient, then f belongs to $\hat{\mathcal{E}}(U)$. (In this case, every proper noncompact face of $\Gamma_+(f)$ can be expressed by the intersection of $\Gamma_+(f)$ and some coordinate hyperplane. Therefore, (iii), (v) in Remark 2.9 imply this assertion.)
- In the one-dimensional case, every nonflat smooth function belongs to $\hat{\mathcal{E}}(U)$. (This is a particular case of the above convenient case.)
- The Denjoy-Carleman (quasianalytic) classes are contained in $\hat{\mathcal{E}}(U)$. (See Proposition 6.10 in [15].)

Unfortunately, the algebraic structure of $\hat{\mathcal{E}}(U)$ is poor. Indeed, it is not closed under addition. For example, consider $f_1(x_1, x_2) = x_1 + x_1 \exp(-1/x_2^2)$ and $f_2(x_1, x_2) = -x_1$. Indeed, both f_1 and f_2 belong to $\hat{\mathcal{E}}(U)$, but $f_1 + f_2 (= \exp(-1/x_2^2))$ does not belong to $\hat{\mathcal{E}}(U)$.

3. Earlier studies

In this section, we state the results of Varchenko [23] and their generalizations [2] relating to the behavior of the oscillatory integral $I(t;\varphi)$ in (1.1). Moreover, we explain some earlier results [24],[1],[20],[18] of the asymptotic behavior of weighted oscillatory integrals.

Throughout this section, the following three conditions are assumed: Let U be an open neighborhood of the origin in $\mathbb{R}^n$.

(A) f is a nonflat smooth (C^∞) function defined on U satisfying that $f(0) = 0$ and $\nabla f(0) = 0$;

(B) g is a nonflat smooth function defined on U;
(C) φ is a smooth function whose support is contained in U.

3.1. **Results of Varchenko.** Let us recall a part of famous results due to Varchenko in [23] and Arnold, Gusein-Zade and Varchenko [1] in the case when f is real analytic on U and $g \equiv 1$. These results require the following condition.

(D) f is real analytic on U and is *nondegenerate* over $\mathbb{R}$ with respect to the Newton polyhedron $\Gamma_+(f)$, i.e., for every compact face γ of $\Gamma_+(f)$, the γ-part f_γ satisfies

$$\nabla f_\gamma = \left(\frac{\partial f_\gamma}{\partial x_1}, \ldots, \frac{\partial f_\gamma}{\partial x_n}\right) \neq (0, \ldots, 0) \quad \text{on the set } (\mathbb{R} \setminus \{0\})^n. \tag{3.1}$$

Theorem 3.1 ([23],[1]). *If f satisfies the condition (D), then the following hold (see Remark 2.7):*

(i) *The progression $\{\alpha\}$ in (1.2) belongs to finitely many arithmetic progressions, which are obtained by using the theory of toric varieties based on the geometry of the Newton polyhedron $\Gamma_+(f)$.*
(ii) $\beta(f, 1) \leq -1/d(f)$;
(iii) *If at least one of the following conditions is satisfied:*
  (a) $d(f) > 1$;
  (b) *f is nonnegative or nonpositive on U;*
  (c) *$1/d(f)$ is not an odd integer and f_{τ_*} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$, where τ_* is the principal face of $\Gamma_+(f)$,*
  then $\beta(f, 1) = -1/d(f)$ and $\eta(f, 1) = m(f)$.

Remark 3.2. Let us consider the case when the phase satisfies a weaker regularity condition:

(E) f belongs to the class $\hat{\mathcal{E}}(U)$ and is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron.

It is shown in [15] that $I(t; \varphi)$ also has an asymptotic expansion of the form (1.2) in the case when the phase satisfies the above condition. Furthermore, Varchenko's results can be directly generalized to the case when the phase belongs to the class $\hat{\mathcal{E}}(U)$. In [15], more precise results are obtained.

Some kind of restrictions to the regularity of the phase, for example the condition: $f \in \hat{\mathcal{E}}(U)$, is necessary in the above results. Indeed, consider the following two-dimensional example: $f(x_1, x_2) = x_1^2 + e^{-1/|x_2|^\alpha}$ $(\alpha > 0)$ and $g \equiv 1$, which is given by Iosevich and Sawyer in [11]. Note that the above f satisfies the nondegeneracy condition (3.1) but it does not belong to $\hat{\mathcal{E}}(U)$. It is easy to see the following: $d(f) = 2$, $m(f) = 1$, $f_{\tau_*}(x_1, x_2) = x_1^2$. It is shown in [11] that $|I(t; \varphi)| \leq Ct^{-1/2}(\log t)^{-1/\alpha}$ for $t \geq 2$. In particular, we have $\lim_{t \to \infty} t^{1/2} I(t; \varphi) = 0$. The pattern of an asymptotic expansion of $I(t; \varphi)$ in this example might be different from that in (1.2).

3.2. **Weighted case.** The following theorem naturally generalizes the assertion (ii) in Theorem 3.1.

Theorem 3.3 ([2]). *Suppose that* (i) f *satisfies the condition* (D) *and* (ii) *at least one of the following conditions is satisfied:*

(a) f *is convenient;*
(b) g *is convenient;*
(c) g *is real analytic on* U;
(d) g *is expressed as* $g(x) = x^p \tilde{g}(x)$ *on* U, *where* $p \in \mathbb{Z}_+^n$ *and* $\tilde{g}$ *is a smooth function defined on* U *with* $\tilde{g}(0) \neq 0$.

Then, we have $\beta(f, g) \leq -1/d(f, g)$.

The following theorem partially generalizes the assertion (iii) in Theorem 3.1.

Theorem 3.4 ([2]). *Suppose that* (i) f *satisfies the condition* (D), (ii) *at least one of the following two conditions is satisfied:*

(a) f *is convenient and* g_{γ_*} *is nonnegative or nonpositive on* U *for all principal faces* γ_* *of* $\Gamma_+(g)$;
(b) g *is expressed as* $g(x) = x^p \tilde{g}(x)$ *on* U, *where every component of* $p \in \mathbb{Z}_+^n$ *is even and* $\tilde{g}$ *is a smooth function defined on* U *with* $\tilde{g}(0) \neq 0$

and (iii) *at least one of the following two conditions is satisfied:*

(c) $d(f, g) > 1$;
(d) f *is nonnegative or nonpositive on* U.

Then the equations $\beta(f, g) = -1/d(f, g)$ *and* $\eta(f, g) = m(f, g)$ *hold.*

Remark 3.5. Similar results to the above two theorems have been obtained in [24], [1], [18]. Vassiliev [24] consider the case when g is a monomial. Okada and Takeuchi [18] consider the case when f is convenient. In our language, the results in [1] can be stated as follows:

(Theorem 8.4 in [1], p. 254) *If* f *is real analytic and is nondegenerate over* $\mathbb{R}$ *with respect to its Newton polyhedron, then*

(i) $\beta(f, g) \leq -1/d(f, g)$;
(ii) *If* $d(f, g) > 1$ *and* $\Gamma_+(g) = \{p\} + \mathbb{R}_+^n$ *with* $p \in \mathbb{Z}_+^n$, *then* $\beta(f, g) = -1/d(f, g)$.

Unfortunately, more additional assumptions are necessary to obtain the above assertions (i), (ii). Indeed, consider the following two-dimensional example:

$$f(x_1, x_2) = x_1^4; \quad g(x_1, x_2) = x_1^2 x_2^2 + e^{-1/x_2^2}.$$

It follows from easy computations that this example violates (i), (ii). (See Section 7.2 in [2].)

Note that some conditions in the assumptions of the above theorems can be considered as typical cases of the assumptions in our new theorems in Section 4, so they are sometimes more useful for the practical applications.

Remark 3.6. Pramanik and Yang [20] obtain a similar result relating to the above equation "$\beta(f,g) = -1/d(f,g)$" in the case when the dimension is two and the weight has the form $g(x) = |h(x)|^{\epsilon}$, where h is real analytic and ϵ is positive. Their approach is based on the Puiseux series expansions of the roots of f and h, which is inspired by the work of Phong and Stein [19]. Their definition of *Newton distance*, which is different from ours, is given through the process of a good choice of coordinate system. As a result, their result does not need the nondegeneracy condition on the phase.

The following theorem shows an interesting "symmetry property" with respect to the phase and the weight.

Theorem 3.7 ([2]). *Suppose that f, g satisfy the condition (D) and that they are convenient and nonnegative or nonpositive on U. Then we have $\beta(x^1 f, g)\beta(x^1 g, f) \geq 1$, where $x^1 = x_1 \cdots x_n$. Moreover, the following two conditions are equivalent:*

(i) $\beta(x^1 f, g)\beta(x^1 g, f) = 1$;

(ii) *There exists a positive rational number d such that $\Gamma_+(x^1 f) = d \cdot \Gamma_+(x^1 g)$.*

If the condition (i) *or* (ii) *is satisfied, then we have $\eta(x^1 f, g) = \eta(x^1 g, f) = n$.*

Lastly, we comment on significance for the investigation in the weighted case. Since the weighted case may be considered as a special case of the unweighted case, unweighted results concerned with the upper bound estimates for oscillation index are also available in the weighted case. However, these estimates are "uniformly" satisfied with respect to the amplitude. Hence, we may obtain more precise results in the case of a specific amplitude.

4. Main results

In this section, our new results in [16] are given. Understanding the resolution of singularities for the phase and the weight deeply, we can generalize and improve the results in [2]. Furthermore, the theorems can be stated in more clear form by using the class $\hat{\mathcal{E}}(U)$, which means that properties of $\hat{\mathcal{E}}(U)$ play crucial roles in the sufficient condition on the phase and the weight. See also [14].

Throughout this section, the three conditions (A), (B), (C) at the beginning of Section 3 are assumed, where U is an open neighborhood of the origin in $\mathbb{R}^n$.

First, let us give a sharp estimate for $I(t;\varphi)$. Since the class $\hat{\mathcal{E}}(U)$ contains many kinds of smooth functions as in Section 2.5, the following theorem generalizes and improves Theorem 3.3.

Theorem 4.1 ([16]). *Suppose that* (i) *f satisfies the condition (E) (see Remark* 3.2) *and* (ii) *at least one of the following two conditions is satisfied:*

(a) *g belongs to the class $\hat{\mathcal{E}}(U)$;*

(b) *f is convenient.*

If the support of φ is contained in a sufficiently small neighborhood of the origin, then there exists a positive constant $C(\varphi)$ independent of t such that

$$|I(t;\varphi)| \leq C(\varphi) t^{-1/d(f,g)} (\log t)^{m(f,g)-1} \quad \textit{for } t \geq 2.$$

In particular, we have $\beta(f,g) \leq -1/d(f,g)$.

Next, let us consider the case when the equality $\beta(f,g) = -1/d(f,g)$ holds. The following theorem generalizes and improves Theorem 3.4.

Theorem 4.2 ([16]). *Suppose that the conditions* (i), (ii) *in Theorem* 4.1 *are satisfied,* (iii) *there exists a principal face* γ_* *of* $\Gamma_+(g)$ *such that* g_{γ_*} *is nonnegative or nonpositive on* U *and* (iv) *at least one of the following three conditions is satisfied:*

(a) $d(f,g) > 1$;
(b) f *is nonnegative or nonpositive on* U;
(c) $1/d(f,g)$ *is not an odd integer and* f_{τ_*} *does not vanish on* $U \cap (\mathbb{R} \setminus \{0\})^n$ *where* τ_* *is a principal face of* $\Gamma_+(f)$ *associated to* γ_* *in* (iii).

Then the equations $\beta(f,g) = -1/d(f,g)$ *and* $\eta(f,g) = m(f,g)$ *hold.*

Remark 4.3. In [16], we give explicit formulae for the coefficient of the leading term of the asymptotic expansion (1.2) under the assumptions (i)-(iii). These explicit formulae show that the above coefficient essentially depends on the principal face-parts f_{τ_*} and g_{γ_*}. The above (i)-(iv) are sufficient conditions for the nonvanishing of the leading term.

Finally, Theorem 3.7 can be generalized in the following form.

Theorem 4.4 ([16]). *Suppose that* f, g *satisfy the condition* (E) *and that they are nonnegative or nonpositive on* U. *Then we have* $\beta(x^{\mathbf{1}}f,g)\beta(x^{\mathbf{1}}g,f) \geq 1$. *Moreover, the following two conditions are equivalent:*

(i) $\beta(x^{\mathbf{1}}f,g)\beta(x^{\mathbf{1}}g,f) = 1$;
(ii) *There exists a positive rational number* d *such that* $\Gamma_+(x^{\mathbf{1}}f) = d \cdot \Gamma_+(x^{\mathbf{1}}g)$.

If the condition (i) *or* (ii) *is satisfied, then we have* $\eta(x^{\mathbf{1}}f,g) = \eta(x^{\mathbf{1}}g,f) = n$.

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